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Controllability and Stabilization in Elasticity, Heat Conduction and Thermoelasticity: Review of Recent Developments*

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Abstract. The aim of this paper is to review developments in exact and approximate controllability as well as stabilization of elastic, thermoelastic, and thermo-viscoelastic bodies. Heat equations are also discussed.

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Abbreviations: HUM – Hilbert Uniqueness Method; RHUM – Reachabillty Hilbert Uniqueness Method

1. Introduction

Exact or at least approximate controllability and stabilization problems are important in the analysis of many engineering systems. One can distinguish three essential topics which naturally arise: rigorous mathematical studies, approximation methods and technical realization of controls. This review is focused on the first two topics. For technical aspects the reader is referred to Armstrong-Hélouvry et al. (1994), Banks (1975), Banks et al. (1996), Benjeddou et al. (1997, 1999), Destuynder et al. (1992), Holnicki-Szulc and Rodellar (1999), Tani et al. (1998), Trindade et al. (1998), and Zhou and Tzou (2000).

We presume that the reader is familiar, more or less, with the books by Komornik (1994b), Lagnese (1989), Lagnese and Lions (1988) and Lions (1988a,b).

An excellent review paper by Russel (1978) still serves as a good introduction to mathematical aspects of controllability, observability and stabilization of systems described mainly by partial differential equations, cf. also Komornik (1988, 1989, 1995, 1997), Nikolski (1998), Ralston (1982).

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* This paper is dedicated to the memory of Professor P.D. Panagiotopoulos

Bardos et al. (1992) formulated the following general principle: “To control, observe, or stabilize solutions of hyperbolic equations, it is necessary that we observe or control at least one point of each ray of geometric optics”. These authors extended this principle to space dimensions higher than one. In higher dimensions, however, the analysis is not as elementary as in dimension one. Advantages and disadvantages of the proposed approach were also discussed.

The books by Avdonin and Ivanov (1989, 1995) offer a systematic approach to the method of moments in controllability problems including applications to one-dimensional problems of elasticity (strings, membranes).

Lebeau (1992) generalized the results due to Bardos, Lebeau and Rauch (Appendix 2 in Lions, 1988a) to Schrödinger type equations and simplified equation of isotropic Kirchhoff plate. This approach exploits microlocal analysis.

Datko (1991) and Datko and You (1991) studied the influence of small time delays on stabilization.

Klamka (1998) studied the approximate controllability of semilinear infinite-dimensional second-order dynamical systems with damping. The general theory was applied to a nonlinear beam equation. The same author (Klamka, 1999) introduced the notion of *approximate positive controllability* for linear infinite-dimensional dynamic systems, for which the controls are taken to be nonnegative. The theory proposed was applied to one-dimensional heat equation.

Lasiecka (1988) investigated the problem of local uniform stability of hyperbolic and parabolic equations with nonlinearities appearing in the boundary conditions (in feedback form).

Bashirov and Kerimov (1997) introduced the controllability notions for partially observed stochastic systems and discussed their relation with complete (exact) and approximate controllabilities. A class of ergodic control problems was investigated in Camilli (1996).

For impulse control problems the reader is referred to Camilli and Falcone (1999) and the references therein.

The presentation in several papers just briefly reviewed is rather abstract. The main aim of our contribution is to review more practical results obtained in the last decade. We focus on papers related to mechanical problems: elasticity, thermoelasticity and thermoviscoelasticity. Heat equations have also been discussed.

In a separate paper a comprehensive review related to a wide class of physical and mechanical problems will be presented (Telega and Bielski, 2000). This class will also comprise wave and wave-like equations, Maxwell’s equations, Stokes and Navier-Stokes equations, KdV equation, fluid-structure systems, structures (beams, membranes, plates and shells), junctions and asymptotic problems, including homogenization.

2. Linear elasticity

Lions (1988a, Chap. 4) applied the method HUM to solve the problem of exact controllability of linear elastic bodies made of homogeneous and isotropic materials. Both Dirichlet and Neumann boundary controls were examined. An extension to homogeneous and anisotropic materials was studied by Telega and Bielski (1996). Microperiodically inhomogeneous bodies were considered in Telega and Bielski (1999). The same problem was later considered by Alabau and Komornik (1997) for the domain Ω in $\mathbb{R}^3$ being a ball of radius R whilst arbitrary bounded and regular domains were assumed in Alabau and Komornik (1998).

In this section we shall present recent results on controllability and stabilization of linear elasticity systems, including numerical realization of the HUM. Earlier results have obviously been discussed in the papers presented below.

We observe that the available results are confined to geometrically linear problems (the theory of small displacements). Geometrically non-linear problems concern only plates and shells, cf. Telega and Bielski (2000). Recent ideas due to Russell (1997) and Renardy and Russell (1999), though not directly related to the subject of this paper, are also worthy of mention. Russell (1997) introduced the notion *formation theory*, which refers to the controlled modification of the geometric configuration, or *shape* of an elastic body by means of attached or embedded actuators. According to Renardy and Russell (1999), the subject material of formation theory concerns the relationships between the applied controls, the actuator distribution and the resulting deformation of the structure. In the above two papers only static, linear elastic problems were investigated. It seems that this emerging theory may find applications in optimal design and bone remodelling. In the last case one can envisage nonmechanical controls. After an extension to quasi-static problems the formation theory may throw new light on the theory of adaptive elasticity, cf. Telega and Lekszycki (2000).

2.1. TRANSMISSION PROBLEM

Laguese (1997) generalized a transmission problem considered by Lions (1988a, Chap. 6) for two wave equations to the case of anisotropic elasticity. More precisely, following Lagnese (1997) we shall consider the exact Dirichlet boundary controllability for such a problem.

Let Ω, Ω_1 be bounded, open, connected sets in $\mathbb{R}^n$ (in practice n = 1,2 or 3) with smooth boundaries Γ and Γ_1, respectively, such that $\bar{\Omega}_1 \subset \Omega$. We set $\Omega_2 = \Omega \backslash \bar{\Omega}_1$, $\Gamma_2 = \partial\Omega_2$; obviously $\Gamma_2 = \Gamma \cup \Gamma_1$. This assumption on domains precludes the case of elastic body made of two layers. Two linear elastic bodies are identified with $\bar{\Omega}_1$ and $\bar{\Omega}_2$. Their elastic moduli $a^{(\alpha)}_{ijkl} (\alpha = 1, 2; i, j, k, l = 1, ..., n)$ satisfy the usual symmetry conditions

$$a^{(\alpha)}_{ijkl} = a^{(\alpha)}_{jikl} = a^{(\alpha)}_{klij}, \tag{2.1}$$

and the following ellipticity condition

$$\exists c_0 > 0, \forall \boldsymbol{E} \in \mathbb{E}^n_s, \quad a^{(\alpha)}_{ijkl} E_{ij} E_{kl} \geqslant c_0 |E|^2. \tag{2.2}$$

Here $\mathbb{E}^n_s$ denotes the space of all real symmetric $n \times n$ matrices.

Lagnese (1997) considered also weaker assumptions on $a^{(\alpha)}_{ijkl}$, where instead of (2.1) we only have

$$a^{(\alpha)}_{ijkl} = a^{(\alpha)}_{klij}, \quad \alpha = 1, 2; i, k = 1, \dots, m; j, l = 1, \dots, n.$$

However, this case is not interesting for the classical linear elasticity. The summation convention over repeated indices is used, unless otherwise stated.

Consider the following *problem of transmission*

$$\begin{cases} \ddot{u}_{1i} - a^{(1)}_{ijkl} u_{1k,lj} = 0 & \text{in } Q_1 = \Omega_1 \times (0, T), \\ \ddot{u}_{2i} - a^{(2)}_{ijkl} u_{2k,lj} = 0, & \text{in } Q_2 = \Omega_2 \times (0, T); \end{cases} \tag{2.3}$$

$$\boldsymbol{u}_2 = \boldsymbol{v}, \text{ on } \Sigma = \Gamma \times (0, T), \tag{2.4}$$

$$\boldsymbol{u}_1 = \boldsymbol{u}_2, a^{(1)}_{ijkl} e_{kl}(\boldsymbol{u}_1) n_j = a^{(2)}_{ijkl} e_{kl}(\boldsymbol{u}_2) n_j, \text{ on } \Sigma_1 = \Gamma_1 \times (0, \Gamma), \tag{2.5}$$

$$\boldsymbol{u}_\alpha(x, 0) = \dot{\boldsymbol{u}}_\alpha(x, 0) = \boldsymbol{0}, \text{ in } \Omega_\alpha. \tag{2.6}$$

Here $\boldsymbol{n} = (n_i)$ is the unit normal to Γ_1 pointing into Ω_1 and

$$e_{ij}(\boldsymbol{u}) = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})/2. \tag{2.7}$$

Obviously, in (2.4) the function $\boldsymbol{v}$ is a control function. The density $\varrho_\alpha (\alpha = 1, 2)$ does not appear in (2.3). It is either incorporated into $\boldsymbol{a}^{(\alpha)}$ or one simply puts $\varrho_\alpha = 1$. Such an assumption is unessential.

Let us define

$$A_\alpha = \inf_{e \in \mathbb{E}^n_s, e \neq 0} \frac{a^{(2)}_{ijkl} E_{ij} E_{kl}}{E_{mp} E_{mp}}. \tag{2.8}$$

The main result of Lagnese is summarized as follows.

THEOREM 1. *Assume that Γ_1 is star-shaped with respect to some point $x^0 \in \Omega_1$ and let*

$$\Gamma(x^0) = \{x \in \Gamma | (x - x_0) \cdot \boldsymbol{n} > 0\}, \Sigma(x^0) = \Gamma(x^0) \times (0, \Gamma), \tag{2.9}$$

$$R(x^0) = \max_{x \in \bar{\Omega}_2} |x - x^0|, \tag{2.10}$$

where $\boldsymbol{n}$ *is the unit outer normal to* Γ. *Let*

$$\begin{aligned}\mathcal{V}_T \; = \; & \{(\boldsymbol{u}_1(\cdot, T), \boldsymbol{u}_2(\cdot, T), \dot{\boldsymbol{u}}_1(\cdot, T), \dot{\boldsymbol{u}}_2(\cdot, T) | \boldsymbol{v} \in L^2(\Sigma)^n, \\ & \boldsymbol{v} = \boldsymbol{0} \text{ on } \Sigma \backslash \Sigma(x^0)\}.\end{aligned}$$

If

$$\forall \boldsymbol{e} \in \mathbb{E}_s^n, a_{ijkl}^{(1)} E_{ij} E_{kl} \geqslant a_{ijkl}^{(2)} E_{ij} E_{kl}, \tag{2.11}$$

and if

$$T > T(x^0) = \frac{2\sqrt{2} R(x^0)}{\sqrt{A_2}}, \tag{2.12}$$

then $\mathcal{V}_T = H \times V^*$ *where*

$$H = L^2(\Omega_1)^n \times L^2(\Omega_2)^n,$$
$$V = \{(\varphi_1, \varphi_2) \in H^1(\Omega_1)^n \times H^1(\Omega_2)^n | \varphi_{2|\Gamma} = \boldsymbol{0}, \varphi_{1|\Gamma_!} = \varphi_{2|\Gamma_1}\}. \qquad \square$$

The proof uses classical multipliers to derive a priori estimates and HUM.

REMARK 1. (i) Theorem 1 can be extended to the situation involving Ω and $p \geqslant 2$ nested open sets $\omega_1, \ldots, \omega_p$ with $\bar{\omega}_i \subset \omega_{i+1}, i = 1, \ldots, p-1, \omega_p = \Omega$. Set: $\Omega_1 = \omega_1, \Omega_i = \omega_i \backslash \bar{\omega}_{i-1}, i = 1, \ldots, p$; $\Gamma_i = \partial \omega_i$. The boundary $\Gamma_i, i = 1, \ldots, p-1$, is star-shaped with respect to a point $x^0 \in \Omega_1$.

(ii) It is not known whether the monotonicity condition (2.11) is necessary for exact controllability in dimension $n \geqslant 2$.

(iii) For isotropic materials the monotonicity condition (2.11) is satisfied provided that

$$\mu_1 \geqslant \mu_2 \text{ and } \lambda_1 \geqslant \lambda_2$$

where $\mu_\alpha, \lambda_\alpha (\alpha = 1, 2)$ are the Lamé coefficients.

(iv) Nicaise (1993) studied the Dirichlet-Neumann boundary controllability of isotropic homogeneous elastic bodies identified with $\bar{\Omega}$, where Ω is a polygonal domain of the plane or a polyhedral domain of the space. Primarily, in Nicaise (1992), the regularity of solutions was examined for both $n = 2$ (corners) and $n = 3$ (vertex and edge singularities). The results of Nicaise (1993) extend those obtained earlier by Grisvard (1989) for the wave equation.

2.2. APPROXIMATE CONTROLLABILITY BY MEANS OF PLANAR BODY FORCES

Consider the following system of linear isotropic elasticity, cf. Zuazua (1996),

$$\begin{aligned} \ddot{\boldsymbol{u}} - \mu \Delta \boldsymbol{u} - (\lambda + \mu) \nabla \text{div} \boldsymbol{u} = \boldsymbol{f} \chi_{\mathfrak{O}}, & \qquad \text{in } Q = \Omega \times (0, T), \\ \boldsymbol{u} = \boldsymbol{0}, & \qquad \text{on } \Gamma \times (0, \Gamma), \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1, & \qquad \text{in } \Omega, \end{aligned} \tag{2.13}$$

where λ, μ denote the Lamé coefficients. Here $\mathfrak{O}$ is an open and nonempty subset of Ω. We assume that $\boldsymbol{f} \in L^2(Q)^n (n = 2, 3)$ is of the form

$$\boldsymbol{f} = (f_1, \ldots, f_{n-1}, 0). \tag{2.14}$$

Prior to the formulation of the approximate controllability result we have to introduce indispensible notations. Let

$$T(\Omega) = \frac{2\delta_n(\Omega; \mathfrak{O})}{\sqrt{\mu}}, \tag{2.15}$$

the quantity δ_n being defined as follows. For any open subset Ω_1 of Ω

$$\delta_n(\Omega; \Omega_1) := \sup_{x \in \Omega \setminus \Omega_1} \inf_{\gamma \in \xi(x;\Omega_1)} l(\gamma), \tag{2.16}$$

where $\xi(x; \Omega_1)$ denotes the set of curves in Ω joining x and $\bar{\Omega}_1$ and $l(\cdot)$ stands for the length of the curve. We set $\delta_n(\Omega; \emptyset) = \infty$.

By $\Omega^{n-1} \subset \mathbb{R}^{n-1}$ and $\Omega^1 \subset \mathbb{R}$ we denote, respectively, the projections of Ω on the hyperplane $x_n = 0$ and on the axis Ox_n. Furthermore, by $\mathfrak{U}^{n-1} \subset \mathbb{R}^{n-1}$ (resp. $\mathfrak{U}^1 \subset \mathbb{R}$) we denote the union of the projections on the hyperplane $x_n = 0$ (resp., on the axis Ox_n) of all those components of the boundary Γ that can be written in the form $x_n = h(x_1, \ldots, x_{n-1})$ with h of class C^2 and such that

$$|\nabla' h(x_1, \ldots, x_{n-1})|^2 \neq \frac{\lambda + 2\mu}{\mu}$$

or

$$\Delta' h(x_1, \ldots, x_{n-1}) \neq 0.$$

By ∇' and Δ' we denote the gradient and Laplacian in the variables $(x_1, \ldots, x_{n-1})$.

The approximate controllability result proved by Zuazua (1996) is formulated as follows.

THEOREM 2. *Let Ω satisfies the following four conditions:*

(i) Ω is a piecewise C^2-bounded domain.
(ii) Some open and nonempty C^2 component of Γ can be written in the form: $x_n = h(x_1, \ldots, x_{n-1})$ with $|\nabla' h|^2 \neq (\lambda + 2\mu)/\mu$ or $\Delta' h \neq 0$ everywhere on that component.
(iii) There exists a point of a C^2 component of the boundary of Ω where the tangent hyperplane to Ω exists, and it is parallel to the axis Ox_n.
(iv) When $n = 3$, either
(iv)$_1$ an open subset of Γ is contained in a plane of the form $x_3 = c$
or
(iv)$_2$ Ω is not symmetric with respect to a plane of the form $x_3 = c$.

Then, if

$$T > 2\frac{\delta_n(\Omega; \mathfrak{O})}{\sqrt{\mu}} + T^*(\Omega),$$

system (2.13) is approximately controllable at time T under the constraint (2.14), where

$$T^*(\Omega) = 2\, min(\frac{1}{\sqrt{\mu}}\delta_{n-1}(\Omega^{n-1}; \mathfrak{U}^{n-1})), \frac{1}{\sqrt{\lambda + 2\mu}}\delta_1(\Omega^1; \mathfrak{U}^1)).$$

More precisely, for all $(\boldsymbol{u}^0, \boldsymbol{u}^1)$ *and* $(\boldsymbol{u}_T^0, u_T^1)$ *in* $H_0^1(\Omega)^n \times L^2(\Omega)^n$ *and* $\epsilon > 0$ *there exists* $\boldsymbol{f} \in L^2(Q)^n$ *obeying (2.14) such that the solution of (2.13) satisfies*

$$[\|\boldsymbol{u}(T) - \boldsymbol{u}_T^0\|^2_{H_0^1(\Omega^n)} + \|\dot{\boldsymbol{u}}(T) - \boldsymbol{u}_T^1\|^2_{L^2(\Omega^n}]^{1/2} \leqslant \epsilon. \qquad \square$$

REMARK 2. (a) Without the constraint (2.14), exact controllability with $L^2(Q)^n$-controls holds for a certain class of $\mathfrak{O}$'s, cf. Lions (1988a). For instance, if $\mathfrak{O}$ is a neighbourhood of the boundary of Ω the exact controllability holds with $T(\Omega) = \text{diam}(\Omega\backslash\mathfrak{O})/\sqrt{\mu}$.
(b) Zuazua (1996) constructed a two-dimensional domain for which $T^*(\Omega) > 0$. This author provided also two examples of noncontrollability.
(c) Under the constraint (2.14), the approximate controllability cannot be obtained directly from Holmgren's uniqueness theorem. Zuazua (1996) solved the problem of uniqueness of the corresponding homogeneous (forward) system by reducing the proof to uniqueness result for scalar wave equations.

2.3. STABILIZATION OF LINEAR ELASTIC BODIES

Earlier results on boundary stabilization of three-dimensional linear elastodynamic system are due to Lagnese (1983). The same author studied also the case of plane strain (two-dimensional elasticity), cf. Lagnese (1991). In both cases the elastic bodies are made of homogeneous isotropic materials.

The aim of the present section is to present the results obtained afterwards by other authors. The papers by Lagnese (1983, 1991), however, largely influenced on the developments which followed.

2.3.1. *Asymptotic Stability of Isotropic Bodies with Internal Damping*

Aassila (1998a) extended the approach used by him for the damped wave equation to the case of homogeneous isotropic geometrically linear elastic bodies, cf. Aassila (1998b).

Consider the following elasticity system with internal damping

$$\begin{array}{ll} \ddot{\boldsymbol{u}} - \mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla\boldsymbol{u} + \boldsymbol{G}(\dot{\boldsymbol{u}}) = \boldsymbol{0}, & \text{in } \Omega\times\mathbb{R}^+, \\ \boldsymbol{u} = \boldsymbol{0}, & \text{on } \Gamma_0\times\mathbb{R}^+, \\ \mu\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}} + (\lambda+\mu)(\mathrm{div}\,\boldsymbol{u})\boldsymbol{n} + g_1\boldsymbol{u} = \boldsymbol{0}, & \text{on } \Gamma_1\times\mathbb{R}^+, \\ \boldsymbol{u}(0) = \boldsymbol{u}^0,\ \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1, & \text{in } \Omega. \end{array} \tag{2.17}$$

Here Ω is a bounded open domain in $\mathbb{R}^n$ having a boundary Γ of class C^2, $\{\Gamma_0, \Gamma_1\}$ is a partition of Γ such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $g_1 : \Gamma_1 \to \mathbb{R}^+$ is a continuously differentiable function.

Each component of the vector $\boldsymbol{G}(\dot{\boldsymbol{u}})$ is specified by $g(\dot{\boldsymbol{u}})$. The function g satisfies the following conditions:

(H_1) g is an increasing function of class C^1,
(H_2) $zg(z) > 0$ for all $z \neq 0$,
(H_3) there exists a number $q \geqslant 2$ satisfying $(n-2)q \leqslant 2n$ and two positive constants c_1, c_2 such that

$$c_1|z| \leqslant |g(z)| \leqslant c_2|z|^{q-2} \quad \text{for all } |z| \geqslant 1$$

We observe that no growth condition at the origin is imposed on g and it suffices to assume that Ω is of finite measure (not necessarily bounded). The assumptions imposed on g preclude the possibility of construction of a standard Lyapunov function, which played an important role in the study performed by Lagnese (1983, 1991).

Using the standard nonlinear semigroup theory we conclude that for any given $(\boldsymbol{u}^0, \boldsymbol{u}^1) \in H^1_{\Gamma_0}(\Omega)^n \times L^2(\Omega)^n$ there exists a unique *mild (weak) solution* $\boldsymbol{u} \in C(\mathbb{R}^+, H^1(\Omega)^n) \cap C^1(\mathbb{R}^+, L^2(\Omega)^n)$ and the linear mapping $(\boldsymbol{u}^0, \boldsymbol{u}^1) \to \boldsymbol{u}$ is continuous with respect to these topologies. The space $H^1_{\Gamma_0}$ is defined by

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma_0\}. \tag{2.18}$$

If $\boldsymbol{u}^0 \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega))^n$, $\boldsymbol{u}^1 \in H^1_{\Gamma_0}(\Omega)^n$, and

$$\mu\frac{\partial\boldsymbol{u}^0}{\partial\boldsymbol{n}} + (\lambda+\mu)(\mathrm{div}\boldsymbol{u}^0)\boldsymbol{n} + g_1\boldsymbol{u}^0 = \boldsymbol{0} \text{ on } \Gamma_1,\ g(\boldsymbol{u}^1) \in L^2(\Omega).$$

Then we have the following regularity property

$$\boldsymbol{u} \in C(\mathbb{R}^+, H^2(\Omega)^n) \cap C^1(\mathbb{R}^+, H^1(\Omega)^n) \cap C^2(\mathbb{R}^+, L^2(\Omega)^n).$$

In this case we say that $\boldsymbol{u}$ is a *strong solution*.

The energy of the solution is defined by

$$E(t) = \frac{1}{2}\int_\Omega \left(|\dot{\boldsymbol{u}}|^2 + \mu|\nabla\boldsymbol{u}|^2 + (\lambda+\mu)(\mathrm{div}\boldsymbol{u})^2\right) dx + \frac{1}{2}\int_{\Gamma_1} g_1(x)|\boldsymbol{u}|^2 d\Gamma. \tag{2.19}$$

If $\boldsymbol{u}$ is a strong solution, standard calculation yields, cf. Komornik (1994b)

$$E(S) - E(T) = \int_S^T \int_\Omega \dot{\boldsymbol{u}} \cdot \boldsymbol{G}(\dot{\boldsymbol{u}}) dx dt \tag{2.20}$$

for all $0 \leqslant S < T < +\infty$. The last identity remains valid for all mild solutions by a density argument. By (H_2) we conclude that $E(t)$ is non-increasing.

The strong asymptotic stability result is formulated as follows.

THEOREM 3. *For every solution of the system (2.17) we have*

$$E(t) \to 0 \text{ as } t \to +\infty. \qquad \square$$

REMARK 3. Caution is needed when reading Aassila's paper (1998a) since in his Eq. (1.1) the damping term is a function with values in $\mathbb{R}^3$ and not in $\mathbb{R}^+$. Consequently, one can consider more general damping by assuming that each component of $\boldsymbol{G}$ is not necessarily the same.

2.3.2. *Boundary Stabilization of Isotropic and Anisotropic Linear Elastic Bodies*

Komornik (1994a) and Alabau and Komornik (1998) devised a constructive method applicable to boundary stabilization problems primarily studied by Lagnese (1983, 1991). More precisely, Komornik (1994a) studied *isotropic* linear elastic bodies whilst Alabau and Komornik (1998) investigated *anisotropic* bodies. In both cases the bodies are made of homogeneous materials. It is thus sufficient to present the results contained in Alabau and Komornik (1998). These authors applied suitable dissipative boundary feedbacks. A nonlinear boundary feedback was applied by Martinez (1999). The last author, however, considered linear elastic bodies made of materials with only cubic symmetry.

Consider the following system, cf. Alabau and Komornik (1998),

$$\begin{array}{ll} \ddot{\boldsymbol{u}} - \operatorname{div}\sigma = \boldsymbol{0} & \text{in } \Omega \times \mathbb{R}^+, \\ \boldsymbol{u} = \boldsymbol{0}, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ \sigma \boldsymbol{n} + A\boldsymbol{u} + B\dot{\boldsymbol{u}} = \boldsymbol{0}, & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1, & \text{in } \Omega, \end{array} \tag{2.21}$$

where $\sigma = (\sigma_{ij})$, $i, j = 1, \ldots, n$, is the stress tensor defined by

$$\sigma_{ij} = a_{ijkl} e_{kl}(\boldsymbol{u}). \tag{2.22}$$

The strain tensor is given by (2.7) and the elastic moduli a_{ijkl} satisfy (2.1) and (2.2). In Eq. $(2.21)_3$, A, B are given nonnegative coefficients, for simplicity. One can, however, easily extend the result which follows to the case where A and B are nonnegative functions of class C^1 on Γ_1.

The energy of the solution of (2.21) is given by

$$E(t) = \frac{1}{2} \int_\Omega (|\dot{\boldsymbol{u}}|^2 + \sigma_{ij} e_{ij}(\boldsymbol{u})) dx + \frac{1}{2} \int_{\Gamma_1} A |\boldsymbol{u}|^2 d\Gamma \tag{2.23}$$

and is a nonincreasing function of $t \in \mathbb{R}^+$.

Geometric assumptions are rather restrictive: it is assumed that

$$\Omega = \Omega_1 \backslash \bar{\Omega}_0, \tag{2.24}$$

where Ω_1 is an open ball, say $\Omega_1 = B(x^0, R)$, Ω_0 is a star-shaped domain with respect to x^0 whose closure belongs to Ω_1, and

$$\Gamma_0 = \partial\Omega_0, \quad \Gamma_1 = \partial\Omega_1. \tag{2.25}$$

The case $\Omega_0 = \emptyset$ is not excluded. The following theorem was proved by Alabau and Komornik (1998).

THEOREM 4. *Let the elasticity tensor (a_{ijkl}) satisfy (2.1), (2.2), and let Ω, Γ_0, and Γ_1 be defined by (2.24), (2.25). Given two positive constants A and B with A < c_0/(4R), there exists a positive number ω such that all (weak) solutions of (2.21) satisfy the energy estimate*

$$E(t) \leqslant E(0)e^{1-\omega t}, \tag{2.26}$$

for all $t \geqslant 0$..

If $\Gamma + 0 \neq \emptyset$, then the result holds also for $A = 0$.

We recall that $x^0 \in \mathbb{R}^n$ is arbitrary and fixed, moreover

$$R = \sup\{|x - x^0|, x \in \Omega\}.$$

REMARK 4. (i) The proof of the last theorem is based on a Lyapunov-type method and a new identity which allows to estimate certain boundary integrals.
(ii) The proof can be adapted to domains such that Ω_1 is *close* to a ball.
(iii) The proof of Th. 4 provides an explicit form of ω which involves a constant depending on A and B but not on the choice of the initial data.
(iv) Applying the approach developed in Komornik (1997), Alabau and Komornik (1998) formulated also a general theorem allowing to construct boundary feedback for observable systems which lead to arbitrarily large decay rates. The second theorem applies to all bounded domains of class C^2, choosing, for instance, $\Gamma_0 = \emptyset$ and $\Gamma_1 = \Gamma$.
(v) Liu (1998b, Th. 2.2) improved the result due to Alabau and Komornik (1998) concerning isotropic bodies: the domain Ω may be star-shaped and the assumption on the *function* $A(x)$ in $(2.21)_3$ can really be weakened, as conjectured in the second paper.
(vi) Tcheugoué Tebou (1996) studied the stabilization problem for the system (2.21) in the two-dimensional case. The material is *isotropic*. The following theorem was proved.

THEOREM 5. *Let $k \in L^\infty(\Gamma_1)$ satisfy*

$$k \geqslant \frac{1}{R} \text{ a.e. on } \Gamma_1, \ |k\mathbf{m}| \leqslant 1 \text{ a.e. on } \Gamma_1.$$

Choose the functions A and B by

$$A = \frac{2}{3}\mu(\boldsymbol{m}\cdot\boldsymbol{n})k^2, \quad B = \sqrt{\mu}(\boldsymbol{m}\cdot\boldsymbol{n})k.$$

Then we have

$$E(t) \leqslant [\exp(1 - \sqrt{\mu}t/3R)]E(0) \quad \forall t \geqslant 3R/\sqrt{\mu},$$

where μ is the Lamé constant. □

We recall that

$$\begin{aligned} R &= \sup\{|\boldsymbol{m}(x)|, x \in \Omega\}, \\ \Gamma_1 &= \{x \in \Gamma | \boldsymbol{m}(x)\cdot\boldsymbol{n}(x) > 0\}, \Gamma_0 = \Gamma\backslash\Gamma_1. \end{aligned}$$

REMARK 5. Martinez (1999) investigated a class of nonlinear boundary feedback laws for bodies made of materials with cubic symmetry, cf. Chernykh (1988). Such materials are characterized by *three independent coefficients*. We recall that isotropic materials are described by only two coefficients (the Lamé constants). The boundary feedback law is given by

$$\sigma\boldsymbol{n} + a\boldsymbol{u} + b\mathbf{g}(\dot{\boldsymbol{u}}) = \mathbf{0} \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \tag{2.27}$$

where $a, b : \Gamma_1 \to \mathbb{R}^+$ are two continuously differentiable functions whilst g_i: $\mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that

$$\forall z \in \mathbb{R}, \qquad |g_i(z)| \leqslant 1 + c|z|, \tag{2.28}$$

for some positive constant c. Martinez (1999) proved a uniform stabilization theorem and derived rather precise decay estimates.

REMARK 6. Horn (1998) established an exponential (uniform) decay of solution for the elastodynamic system of isotropic elasticity. Only the velocity feedback is acting through the boundary:

$$\sigma(\boldsymbol{u})\boldsymbol{n} = -\dot{\boldsymbol{u}} \quad \text{on } \Gamma_1 \times \mathbb{R}^+.$$

In our opinion, one should write:

$$\sigma(\boldsymbol{u})\boldsymbol{n} = -\alpha\dot{\boldsymbol{u}}, \quad \alpha > 0, \quad \text{on } \Gamma_1 \times \mathbb{R}^+.$$

The constants α is not dimensionless; particularly one can take $\alpha = 1$ (in appropriate units, depending on the units of the velocity $\dot{\boldsymbol{u}}$ and tractions $\sigma(\boldsymbol{u})\boldsymbol{n}$).

The uniform stability theorem of Horn (1998) does not require the usual strong geometric assumption on Γ_1. Under the usual assumption of smooth boundary of Ω, it suffices to impose the following standard condition:

$$\boldsymbol{m}(x)\cdot\boldsymbol{n} \leqslant 0 \qquad \text{on } \Gamma_0.$$

The uniform stability theorem is based on the multiplier method and sharp trace estimates for the tangential derivative of the displacements on the boundary as well as on the unique continuation results for the corresponding static system.

2.4. NUMERICAL APPROACH TO EXACT BOUNDARY CONTROLLABILITY

Asch and Lebeau (1996) elaborated a numerical approach to solve the exact boundary controllability for the wave equation. Such an approach, based on the conjugate gradient method, was extended by Asch and Vai (1998) to two dimensional *isotropic* elasticity. Here we shall present the main idea in a more general context of anisotropic elasticity, cf. comments given at the beginning of Section 2. The conjugate gradient method can readily be extended to three-dimensional problems.

Let Ω be a bounded, sufficiently regular domain of $\mathbb{R}^2$. Consider the following system of dynamic elasticity

$$\ddot{\boldsymbol{u}} - \operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{0}, \qquad \text{in } Q = \Omega \times (0, T), \tag{2.29}$$

$$\boldsymbol{u}(x_\alpha, 0) = \boldsymbol{u}^0(x), \qquad \dot{\boldsymbol{u}}(x_\alpha, 0) = \boldsymbol{u}^1(x), \quad \text{in } \Omega, \tag{2.30}$$

$$\boldsymbol{u} = \boldsymbol{v}, \qquad \text{on } \Sigma = \Gamma \times (0, T) \tag{2.31}$$

or

$$\boldsymbol{u} = \begin{cases} \boldsymbol{v} \text{ on } \Sigma_0 = \Gamma_0 \times (0, T), \\ \boldsymbol{0} \text{ on } \Sigma \backslash \Sigma_0. \end{cases} \tag{2.32}$$

As usual, we want to find a control $\boldsymbol{v}$ such that if $\boldsymbol{u}$ is a solution to (2.29), (2.30) and (2.31) or (2.32), then

$$\boldsymbol{u}(x_\alpha, T) = \dot{\boldsymbol{u}}(x_\alpha, 0) = \boldsymbol{0} \tag{2.33}$$

Here $x = (x_\alpha)$, $\alpha = 1, 2$, and

$$\sigma_{\alpha\beta}(\boldsymbol{u}) = a_{\alpha\beta\lambda\mu} e_{\lambda\mu}(\boldsymbol{u}). \tag{2.34}$$

The elasticity tensor $a_{\alpha\beta\lambda\mu}$ satisfies (2.1), (2.2), whilst $e_{\alpha\beta}(\boldsymbol{u})$ is given by (2.7).

Let us pass to the characterization of the control $\boldsymbol{v}$. We set

$$V = H_0^1(\Omega)^2 \times L^2(\Omega)^2, \quad V' = H^{-1}(\Omega)^2 \times L^2(\Omega)^2. \tag{2.35}$$

Let $(\boldsymbol{f}^0, \boldsymbol{f}^1) \in V$. First, we solve the following system

$$\begin{aligned} \ddot{\boldsymbol{\Phi}} - \operatorname{div}\boldsymbol{\sigma}(\boldsymbol{\Phi}) &= \boldsymbol{0} \quad \text{in } Q, \\ \boldsymbol{\Phi}(x, 0) = \boldsymbol{f}^0(x), \quad \dot{\boldsymbol{\Phi}}(x, 0) &= \boldsymbol{\Phi}^1(x), \quad \text{in } \Omega, \\ \boldsymbol{\Phi} &= \boldsymbol{0} \quad \text{on } \Sigma. \end{aligned} \tag{2.36}$$

Next, we solve the backward problem

$$\begin{aligned}
&\ddot{\boldsymbol{\Psi}} - \operatorname{div}\sigma(\boldsymbol{\Psi}) = \mathbf{0}, && \text{in } Q,\\
&\boldsymbol{\Psi}(x,T) = \mathbf{0},\ \dot{\boldsymbol{\Psi}}(x,T) = \mathbf{0}, && \text{in } \Omega,\\
&\boldsymbol{\Psi} = \begin{cases} \boldsymbol{\sigma}(\boldsymbol{\Phi})\boldsymbol{n} & \text{on } \Sigma_0,\\ \mathbf{0} & \text{on } \Sigma\setminus\Sigma_0. \end{cases}
\end{aligned} \tag{2.37}$$

Finally, we introduce the linear operator Λ

$$\begin{aligned}
&\Lambda : V \to V'\\
&(\boldsymbol{f}^o, \boldsymbol{f}^1) \to (\dot{\boldsymbol{\Psi}}(x,0), -\boldsymbol{\Psi}(x,0))
\end{aligned} \tag{2.38}$$

It is known that for T sufficiently large, Λ is an isomorphism of V on V', cf. Telega and Bielski (1996), Alabau and Komornik (1998). In the case of isotropy, $T > \frac{2}{\sqrt{\mu}} R(x^0)$ where $x^0 \in \mathbb{R}^2$ and

$$R(x^0) = \max_{x\in\bar{\Omega}} |x - x^0|,$$

$$\Gamma_0 = \Gamma(x^0) = \{x \in \Gamma | (x - x^0)\cdot \boldsymbol{n}(x) > 0\}$$

Thus the method HUM means that we have to find the unique solution $(\boldsymbol{f}^0, \boldsymbol{f}^1) \in V$ such that

$$\Lambda(\boldsymbol{f}^0, \boldsymbol{f}^1) = (\boldsymbol{\Psi}(x,0), -\boldsymbol{\Psi}(x,0)).$$

The bilinear form $\langle \Lambda(\cdot), (\cdot)\rangle$ is symmetric and continuous on $V \times V$. Moreover, under the assumptions ensuring applicability of the method HUM this bilinear form is V-elliptic. Consequently, the Lax-Milgram lemma applies, cf. Yosida (1978) and we can use a conjugate gradient algorithm to solve the following variational problem:

$$\left|\begin{aligned}
&\text{find } (\boldsymbol{f}^0, \boldsymbol{f}^1) \in V \text{ such that}\\
&\langle \Lambda(\boldsymbol{f}^0, \boldsymbol{f}^1), (\tilde{\boldsymbol{f}}^0, \tilde{\boldsymbol{f}}^1)\rangle = \langle \dot{\boldsymbol{\Psi}}(x,0), -\boldsymbol{\Psi}(x,0), (\tilde{\boldsymbol{f}}^0, \tilde{\boldsymbol{f}}^1)\rangle\ \forall (\tilde{\boldsymbol{f}}^0, \tilde{\boldsymbol{f}}^1) \in V
\end{aligned}\right.$$

Conjugate gradient algorithm
The scalar product of V is defined by

$$(\boldsymbol{U}, \boldsymbol{W})_V = \int_\Omega (\nabla \boldsymbol{u}^0 \cdot \nabla \boldsymbol{w}^0 + \boldsymbol{u}^1 \cdot \boldsymbol{w}^1) dx,$$

where $\boldsymbol{U} = (\boldsymbol{u}^0, \boldsymbol{u}^1)$, $\boldsymbol{W} = (\boldsymbol{w}^0, \boldsymbol{w}^1)$.

Step 0. Initialization

- let $(\boldsymbol{f}_0^0, \boldsymbol{f}_0^1) = ((f_0^{0,1}, f_0^{0,2}), (f_0^{1,1}, f_0^{1,2})) \in V$;
- let $(\boldsymbol{g}_0^0, \boldsymbol{g}_0^1) \in V$;

- solve the equation in Φ^0:

$$\begin{aligned} \ddot{\boldsymbol{\Phi}}^0 - \mathrm{div}\boldsymbol{\sigma}(\boldsymbol{\Phi}^0) = \mathbf{0}, &\quad \text{in } Q, \\ \boldsymbol{\Phi}^0(x,0) = \boldsymbol{f}_0^0(x), \quad \dot{\boldsymbol{\Phi}}(x,0) = \boldsymbol{f}_0^1(x), &\quad \text{in } \Omega, \\ \boldsymbol{\Phi}^0 = \mathbf{0}, &\quad \text{on } \Sigma. \end{aligned} \tag{2.39}$$

- solve the backward equation

$$\begin{aligned} &\ddot{\boldsymbol{\Psi}}^0 - \mathrm{div}\boldsymbol{\sigma}(\boldsymbol{\Psi}^0) = \mathbf{0} && \text{in } Q, \\ &\boldsymbol{\Psi}^0(x,T) = \mathbf{0}, \quad \dot{\boldsymbol{\Psi}}^0(x,T) = \mathbf{0}, && \text{in } \Omega, \\ &\boldsymbol{\Psi}^0 = \begin{cases} \boldsymbol{\sigma}(\boldsymbol{\Psi}^0)\boldsymbol{n} & \text{on } \Sigma_0, \\ \mathbf{0} & \text{on } \Sigma\backslash\Sigma_0; \end{cases} \end{aligned} \tag{2.40}$$

- solve

$$\begin{aligned} &-\Delta \boldsymbol{g}_0^0 = \dot{\boldsymbol{\Psi}}^0(x,0) - \boldsymbol{u}^1, && \text{in } \Omega, \\ &\boldsymbol{g}_0^0 = \mathbf{0}, && \text{on } \Gamma, \end{aligned} \tag{2.41}$$

and

$$\boldsymbol{g}_0^1 = \boldsymbol{u}^0 - \boldsymbol{\Psi}^0(x,0) \qquad \text{in } \Omega,$$

- if $(\boldsymbol{g}_0^0, \boldsymbol{g}_0^1) = \mathbf{0}$ or is small, put $(\boldsymbol{f}^0, \boldsymbol{f}^1) = (\boldsymbol{f}_0^0, \boldsymbol{f}_0^1)$ and stop the algorithm; if not, establish the first direction of descent: $(\boldsymbol{w}_0^0, \boldsymbol{w}_0^1) = (\boldsymbol{g}_0^0, \boldsymbol{g}_0^1)$.

Step 1. Descent. For $k \geqslant 0$, suppose that $(\boldsymbol{f}_k^0, \boldsymbol{f}_k^1)$, $(\boldsymbol{g}_k^0, \boldsymbol{g}_k^1)$, and $(\boldsymbol{w}_k^0, \boldsymbol{w}_k^1)$ are known; the new iterates $(\boldsymbol{f}_{k+1}^0, \boldsymbol{f}_{k+1}^1)$, $(\boldsymbol{g}_{k+1}^0, \boldsymbol{g}_{k+1}^1$, and $(\boldsymbol{w}_{k+1}^0, \boldsymbol{w}_{k+1}^1)$, are calculated as follows:

- solve the equation in $\bar{\boldsymbol{\Phi}}_k$:

$$\begin{aligned} \ddot{\bar{\boldsymbol{\Phi}}}_k - \mathrm{div}\boldsymbol{\sigma}(\bar{\boldsymbol{\Phi}}_k) = \mathbf{0} &\quad \text{in } Q, \\ \bar{\boldsymbol{\Phi}}_k(x,0) = \boldsymbol{w}_k^0(x), \quad \dot{\bar{\boldsymbol{\Phi}}}_k(x,0) = \boldsymbol{w}_k^1(x), &\quad \text{in } \Omega, \\ \bar{\boldsymbol{\Phi}}_k = \mathbf{0}, &\quad \text{on } \Sigma; \end{aligned} \tag{2.42}$$

- solve the backward equation

$$\begin{aligned} &\ddot{\bar{\boldsymbol{\Psi}}}_k - \mathrm{div}\boldsymbol{\sigma}(\bar{\boldsymbol{\Psi}}_k) = \mathbf{0}, && \text{in } Q, \\ &\bar{\boldsymbol{\Psi}}_k(x,T) = \mathbf{0}, \quad \bar{\boldsymbol{\Psi}}_k(x,T) = \mathbf{0}, && \text{in } \Omega \\ &\bar{\boldsymbol{\Psi}}_k = \begin{cases} \boldsymbol{\sigma}(\dot{\bar{\boldsymbol{\Psi}}}_k)\boldsymbol{n} & \text{on } \Sigma_0, \\ \mathbf{0} & \text{on } \Sigma\backslash\Sigma_0; \end{cases} \end{aligned} \tag{2.43}$$

- solve

$$\begin{aligned} -\Delta \bar{\boldsymbol{g}}_k^0 &= \dot{\bar{\boldsymbol{\Psi}}}_k(x,0), \quad &\text{in } \Omega, \\ \bar{\boldsymbol{g}}_k^0 &= \mathbf{0}, &\text{on } \Gamma, \end{aligned} \tag{2.44}$$

  and

$$\bar{\boldsymbol{g}}_k^1 = -\bar{\boldsymbol{\Psi}}_k(x,0) \quad \text{in } \Omega;$$

- calculate

$$\varrho_k = \frac{\Sigma_{\alpha=1}^2 \int_\Omega (|\nabla g_k^{0,\alpha}|^2 + |g_k^{1,\alpha}|^2) dx}{\sum_{\alpha=1_\Omega}^2 (\nabla \bar{g}_k^{0,\alpha} \cdot \nabla w_k^{0,\alpha} + \bar{g}_k^{1,\alpha} w_k^{1,\alpha}) dx}$$

  with no summation over k.
- Update all quantities:

$$\begin{aligned} (\boldsymbol{f}_{k+1}^0, \boldsymbol{f}_{k+1}^1) &= (\boldsymbol{f}_k^0, \boldsymbol{f}_k^1) - \varrho_k (\boldsymbol{w}_k^0, \boldsymbol{w}_k^1), \\ \boldsymbol{\Phi}_{k+1} &= \boldsymbol{\Phi}_k - \varrho_k \bar{\boldsymbol{\Phi}}_k, \\ \boldsymbol{\Psi}_{k+1} &= \boldsymbol{\Psi}_k - \varrho_k \bar{\boldsymbol{\Psi}}_k, \\ (\boldsymbol{g}_{k+1}^0, \boldsymbol{g}_{k+1}^1) &= (\boldsymbol{g}_k^0, \boldsymbol{g}_k^1) - \varrho_k (\bar{\boldsymbol{g}}_k^0, \bar{\boldsymbol{g}}_k^1). \end{aligned} \tag{2.45}$$

Obviously, in Eq. (2.45) there is no summation over k.
Step 2. Convergence test and construction of new descent direction:

- if $(\boldsymbol{g}_{k+1}^0, \boldsymbol{g}_{k+1}^1) = \mathbf{0}$ or $(\boldsymbol{g}_{k+1}^0, \boldsymbol{g}_{k+1}^1)$ is small, set $(\boldsymbol{f}^0, \boldsymbol{f}^1) = (\boldsymbol{f}_{k+1}^0, \boldsymbol{f}_{k+1}^1)$, $\boldsymbol{\Phi} = \boldsymbol{\Phi}_{k+1}$, $\boldsymbol{\Psi} = \boldsymbol{\Psi}_{k+1}$ and stop the algorithm;
- if not, establish the new direction of search $(\boldsymbol{w}_{k+1}^0, \boldsymbol{w}_{k+1}^1)$:

$$\gamma_k = \frac{\|(\boldsymbol{g}_{k+1}^0, \boldsymbol{g}_{k+1}^1)\|^2}{\|(\boldsymbol{g}_k^0, \boldsymbol{g}_k^1)\|^2} = \frac{\sum_{\alpha=1}^2 \int_\Omega (|\nabla g_{k+1}^{0,\alpha}|^2 + |g_{k+1}^{1,\alpha}|^2) dx}{\sum_{\alpha=1}^2 \int_\Omega (|\nabla g_k^{0,\alpha}|^2 + |g_k^{1,\alpha}|^2) dx}$$

$$(\boldsymbol{w}_{k+1}^0, \boldsymbol{w}_{k+1}^1) = (\boldsymbol{g}_{k+1}^0, \boldsymbol{g}_{k+1}^1) + \gamma_k (\boldsymbol{w}_k^0, \boldsymbol{w}_k^1), \tag{2.46}$$

and go to Step 1 with $k = k + 1$; the summation convention does not apply to the last step.

Asch and Vai (1998) performed also a discretization of the presented algorithm. It was assumed that Ω is a square of length 1. The approach used is similar to the one developed by Asch and Lebeau (1996) for the wave equation. Various numerical aspects related to the problem studied were carefully discussed. For instance, the *energetic cost vector* and *energetic cost factor* are criterions for the evaluation of the cost of the control and the energetic cost. The results of numerical calculations show the propagation of elastic waves without the control on the boundary and with the boundary control for several cases of control.

3. Heat Equations

The comprehensive papers by Glowinski and Lions (1994, 1996) constitute a good introduction to approximate controllability of diffusion equation and relevant numerical methods, including numerical tests. In this section we review recent developments and topics not discussed by these authors.

3.1.

Fabre et al. (1993) considered the approximate controllability of the following system

$$\begin{aligned} \dot{u} - \Delta u + au = v\chi\mathfrak{O}, \quad & \text{in } Q = \Omega \times (0, T), \\ u = 0, \quad & \text{on } \Sigma = \Gamma \times (0, T), \\ u(0) = 0 \quad & \text{in } \Omega. \end{aligned} \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^n, n > 1$, is an open and bounded set with C^2 boundary and $\mathfrak{O}$ an open and nonempty subset of Ω. As previously, $v = v(x, t)$ represents the control function. We observe that we can consider the initial condition $u(0) = u^0$.. However, since the problem is linear it suffices to treat the case $u^0 = 0$.

System (3.1) is said to be L^p*-approximately controllabl*e in $L^2(\Omega)$ at time $T > 0$ if the following holds:
"the set of reachable states at time $T > 0$,

$$\mathfrak{S}(T) = \{u(x, T)|u \text{ is solution of (3.1) with } L^p(\mathfrak{O} \times (0, T))\}, \tag{3.2}$$

is dense in $L^2(\Omega)$"; here $1 < p \leqslant \infty$. Obviously, this definition is equivalent to the following statement: for every $\epsilon > 0$ and for every $u_T \in L^2(\Omega)$ there exists $v \in L^P(\Omega)$ such that $\|u(T) - u_T\|_{L^2(\Omega)} \leqslant \epsilon$.

By a standard approach we prove that the problem of the approximate controllability reduces to the following uniqueness property: If Φ satisfies the adjoint system:

$$\begin{aligned} -\dot{\Phi} - \Delta\Phi + a\Phi = 0, \quad & \text{in } Q, \\ \Phi = 0, \quad & \text{on } \Sigma, \\ \Phi(T) = \Phi^0 \in L^2(\Omega), & \end{aligned} \tag{3.3}$$

then $\Phi = 0$ in $\mathfrak{O} \times (0, T)$ implies $\Phi = 0$ in $\Omega \times (0, T)$. We recall that since the velocity of heat propagation is infinite, T may be arbitrary small.

Let us pass to a characterization of the control which minimizes the $L^p(\mathfrak{O} \times (0, T))$-norm and particularly when $p = +\infty$. To this end we recall that $g \in sgn(s)$ if $g(x, t) = s(x, t)/|s(x, t)|$ if $s(x, t) \neq 0$ and $|g(x, t)| \leqslant 1$ on the set $Q_1 = \{(x, t)|s(x, t) = 0\}$. By $sgn_0(s)$ we denote the element of $sgn(s)$ which is equal to 0 on Q_1. We will say that a function v is quasi bang-bang if $v(x, t) = \lambda g(x, t)$ where λ is a constant and $g \in sgn(s)$ for some function s.

We set

$$U_{\rm ad}^{p} = \{v \in L^{p}(\mathfrak{O} \times (0,T)) | \text{the solution of (3.1) satisfies} \\ \|u(T) - u_T\|_{L^2(\Omega)} \leqslant \epsilon\}.$$

Fabre et al. (1993) proved the following theorem.

THEOREM 6. *For $2 \leqslant p \leqslant +\infty$, we have*

$$\min\{\frac{1}{2}\|v\|^2_{L^p(\mathfrak{O}\times(0,T))} | v \in U^p_{ad}\} = -\min\{J_p(\Phi^0) | \Phi^0 \in L^2(\Omega)\}, \tag{3.4}$$

where

$$J_p(\Phi^0) = \frac{1}{2}\left(\int_{\mathfrak{O}\times(0,T)} |\Phi(x,t)|^{p'} dxdt\right)^{2/p'} + \epsilon\|\Phi^0\|_{L^2(\Omega)} - \int_\Omega u^1\Phi^0 dx, \tag{3.5}$$

and Φ is the solution of (3.3) with $\Phi(T) = \Phi^0$; moreover $\frac{1}{p} + \frac{1}{p} = 1$.

Also, if v_p denotes the control which minimizes the L^p-norm over $U^p_{\rm ad}$ then $\{v_p\}_{p<+\infty}$ is bounded in $L^2(\mathfrak{O} \times (0,T))$ and if $\bar{v}$ is a limit point of $\{v_p\}_{p<+\infty}$ when p tends to $+\infty$ then

$$\bar{v} \in U^\infty_{\rm ad} \text{ and } \|\bar{v}\|_{L^\infty(\mathfrak{O}\times(0,T))} = \min\{\|v\|_{L^\infty(\mathfrak{O}\times(0,T))} | v \in U^\infty_{\rm ad}\}, \\ \bar{v} \quad \text{is quasi bang-bang}.$$

REMARK 7. (i) If $p < +\infty$, there exists a unique control minimizing the L^p-norm over admissible controls; for $p = +\infty$ the uniqueness result is not available. (ii) If $p < +\infty$, the problem $\min\{J_p(\Phi^0)|\Phi^0 \in L^2(\Omega)\}$ is the dual problem of $\min\{(1/2\|v\|^2_{L^p(\mathfrak{O}\times(0,T))}|v \in U^p_{\rm ad}\}$, in the sense of Rockafellar's theory of duality presented in Ekeland and Temam (1976).

If $p = +\infty$, the problem $\min\{J_\infty(\Phi^0)|\Phi^0 \in L^2(\Omega)\}$ is a primal problem; the dual problem means evaluating

$$\min\{1/2\|v\|_{L^\infty(\mathfrak{O}\times(0,T))} | v \in U^\infty_{ad}\}$$

3.2.

Lebeau and Robbiano (1995) solved the problem of exact internal controllability of the linear heat equation posed on a compact and connected Riemann manifold of class C^∞. This interesting result was inspired by the paper of Russell (1973) who proved that an exact controllability result for the wave equation implies the exact controllability of the heat equation. Technical details of proofs for this particular case were carefully elaborated. The above authors exploited estimates on the elliptic equation $\frac{\partial^2}{\partial t^2} + \Delta$, deduced from Carleman's estimates. We observe that in the parabolic case there is no geometric constraint on the control region.

3.3.

Ji and Lasiecka (1998) studied the following abstract model:

$$\begin{aligned} &\dot{u} = Au + Bv \quad \text{in } [D(A^*)]', \\ &u(0) = u_0 \in H, \\ &y = Cu, \end{aligned}$$

where $[D(A^*]'$ is the dual of $D(A^*)$ with respect to the H-topology, A is a generator of an analytic semigroup defined on a Hilbert space H, B is the control operator, and C is the observation. Both control and observation are modelled by fully unbounded operators. Under certain hypotheses on A, B and C there exists an infinite-dimensional "compensator" z, the solution of

$$\dot{z} = (A + BF - KC)z + KCu,$$

such that the feedback control

$$v = Fz$$

exponentially stabilizes the abstract model considered. The linear operators F, K appear in the stabilizability-detectability assumption.

The study was motivated by recent applications of "smart material" technology in the context of control and stabilization. Smart actuators and sensors such as piezoceramic patches, piezoelectric devices are modelled by unbounded operators, cf. Banks et al. (1996).

The main result of Ji and Lasiecka (1998) provides a construction of the partially observed stabilizing feedback. Elaboration of approximating schemes requires many assumptions on approximation of operators A, B, C, F and K.

The general approach to partially observed control systems just sketched was used by Ji and Lasiecka (1998) to the following heat equation with boundary control and boundary observation:

$$\begin{aligned} \dot{u} = \Delta u + c^2 u, &\quad \text{in } \Omega \times \mathbb{R}^+, \\ u = v, &\quad \text{on } \Gamma \times \mathbb{R}^+, \\ y = \frac{\partial \mathbf{u}}{\partial \mathbf{n}}, &\quad \text{in } \Gamma \times \mathbb{R}^+, \\ u(0) = u^0, &\quad \text{in } \Omega. \end{aligned}$$

The bounded domain $\Omega \subset \mathbb{R}^n$ is either smooth or convex.

3.4.

Fabre et al. (1995) examined the problem of *approximate controllability of the semilinear heat equation* when the control acts on any open and nonempty subset of $\Omega \subset \mathbb{R}^n$, $n > 1$, or on a part of the boundary. The assumptions on Ω and $\mathfrak{O}$ have been specified in Sec. 3.1. Let f be a real and globally Lipschitz function such that

$$\exists c > 0, \exists d > 0, \quad |f(z)| \leqslant c|z| + d. \tag{3.6}$$

Internal controllability

Consider now the following semilinear heat equation

$$\begin{aligned} \dot{u} - \Delta u + f(u) &= g\chi\mathfrak{O}, \quad \text{in } Q, \\ u &= 0, \quad \text{on } \Sigma, \\ u(0) &= u^0, \quad \text{in } \Omega, \end{aligned} \tag{3.7}$$

where g represents the control function and $\chi\mathfrak{O}$ is the characteristic function of $\mathfrak{O}$, the set where controls are supported.

The definition of approximate controllability is similar to the one given in Sec. 3.1; now, however, u is the solution of (3.7) and $g \in L^p(Q)$. Furthermore, system (3.7) is approximately controllable in $C_0(\Omega)$ (the space of uniformly continuous functions vanishing on $\Gamma = \partial\Omega$, endowed with the norm of the supremum) at time $T > 0$ if for every $u^0 \in C_0(\Omega)$ the set

$$\mathfrak{S}(t) = \{u(x, T) | u \text{ is the solution of (3.7) with } g \in L^\infty(Q)\},$$

is dense in $C_0(\Omega)$.

Equivalently we may formulate these definitions as follows: For every $u_T \in L^p(\Omega)$ (respectively $C_0(\Omega)$) and for every $\epsilon > 0$, there exists a control $g \in L^p(Q)$ (respectively $L^\infty(Q)$) such that the solution u of (3.7) satisfies $\|u(T) - u_T\|_{L^p(\Omega)} \leqslant \epsilon$ (respectively $\|u(T) - u_T\|_{C_o(\Omega) \leqslant \epsilon}$).

The internal approximate controllability result is formulated as follows.

THEOREM 7. *Under the above assumptions on f, system (3.7) is approximately controllable in $L^p(\Omega)$ for $1 \leqslant p < \infty$ and in $C_0(\Omega)$ at any time $T > 0$. Moreover, for every $u^1 \in L^p(\Omega)$ (respectively $C_0(\Omega)$) and for every $\epsilon > 0$, we can find a control $g \in L^p(Q)$ (respectively $L^\infty(Q)$) of the form:*

$$g(x, t) \in \left(\int_{\mathfrak{O} \times (0,T)} |\varphi(x, t)| dx dt \right) sgn(\varphi) \chi_{\mathfrak{O} \times (0,T)},$$

where φ is the solution of a suitable heat equation, such that $\|u(T) - u_T\|_{L^p(\Omega)} \leqslant \epsilon$ (respectively $\|u(T) - u_T\|_{C_0(\Omega)} \leqslant \epsilon$).

The proof combines variational approach to the linear equation with fixed point theorem (Kakutani's theorem).

REMARK 8. (i) Controls of the form appearing in the last theorem are quasi bang-bang controls, cf. Sec. 3.1.
(ii) Fabre et al. (1995) considered also approximate boundary controllability in $L^p(\Omega)$ for any $1 \leqslant p < \infty$ and arbitrary $T > 0$. In this case the system studied is

$$\begin{aligned} \dot{u} - \Delta u + f(u) = 0, &\quad \text{in } Q, \\ u = g_1 \chi_{\Sigma_1}, &\quad \text{on } \Sigma, \\ u(0) = u^0, & \end{aligned} \tag{3.8}$$

where $g_1 = g_1(\sigma, t)$ represents the control function, σ is the boundary variable and χ_{Σ_1} is the characteristic function of $\Sigma_1 = \Gamma_1 \times (0, T)$, the set where the controls are supported.

System (3.8) is said to be approximately controllable in $L^p(\Omega)$ at time $T > 0$ if the following holds: For every $u^0 \in L^p(\Omega)$ the set of reachable states at time $T > 0$

$$\mathfrak{S}_b(T) = \{u(x, T) | u \text{ is the solution of (3.8) with } g_1 \in L^\infty(\Sigma)\}$$

is dense in $L^p(\Omega)$.

3.5.

Zuazua (1997) introduced the notion of *finite* (or finite dimensional) *null controllability* for the semilinear heat equation. This notion is introduced as follows: Given an initial data u^0 in $L^2(\Omega)$ and a control time $T > 0$, find a control $g \in L^2(Q)$ such that the solution of (3.7) satisfies

$$\Pi_{H_{\text{fin}}} u(T) = 0.$$

Here H_{fin} is a finite dimensional subspace of $L^2(\Omega)$ and $\Pi_{H_{\text{fin}}}$ denotes the orthogonal projection from $L^2(\Omega)$ into H_{fin}.

It seems that this notion of controllability is of interest from a computational point of view, since in practice one can only test numerically the reachability of a finite number of constraints.

The function f in (3.7) is of class C^1 and $f(0) = 0$. The control function $g = g(x, t)$ is assumed to be in $L^2(Q)$ and u^0 in $L^2(\Omega)$. Obviously, different functional settings are possible. Under appropriate growth condition on f and if

$$\|u^0\|_{L^2(\Omega)} \leqslant \epsilon, \tag{3.9}$$

then the solution of (3.7) satisfies (3.9); ϵ depends on H_{fin}, T and $\mathfrak{O}$. This statement characterizes the local null controllability.

If f is *globally Lipschitz*, the following controllability result holds.

THEOREM 8. *For any* $T > 0$, $\mathfrak{O}$ *open non-empty subset of* Ω, H_{fin} *finite dimensional subspace of* $L^2(\Omega)$, u^0, $u_T \in L^2(\Omega)$, *and* $\epsilon > 0$, *there exists* $g \in L^2(Q)$ *such that the solution* u *of (3.7) satisfies*

$$\|u(T) - u_T\|_{L^2(\Omega)} \leqslant \epsilon,$$

and

$$\Pi_{H_{\text{fin}}}(u(T)) = \Pi_{H_{\text{fin}}}(u_T).$$

REMARK 9. (i) Theorem 3.3 implies that the approximate control driving the initial data u^0 to the ball $B(u_T, \epsilon)$ of $L^2(\Omega)$ can be chosen so that the final state satisfies simultaneously a finite number of exact constraints.
(ii) Similar results can be obtained for the problem of boundary control.

REMARK 10. Khapalov (1999d) considered *finite exact controllability* and approximate controllability for the *one-dimensional semilinear* heat equation in $Q = (0, 1) \times (0, T)$

$$\begin{aligned} &\dot{u} = u_{xx} + f(u) + v(t)\chi_{(l_1,l_2)}(x),\ v \in L^2(0, T), \\ &u(0, t) = u(1, t) = 0, \quad u(x, 0) = u^0 \in L^2(0, 1), \end{aligned} \tag{3.10}$$

where $(l_1, l_2) \subset (0, 1)$.

The controllability results were proved under the assumption that $l_2 \pm l_1$ are irrational numbers. Since the problems is one-dimensional in space, the proofs combine author's asymptotic method (see Khapalov, 1995) with the Riesz basis approach relevant to the linear boundary problems with pointwise controls.

3.6.

Khapalov (1995) considered *approximate controllability* of the following nonlinear equation

$$\begin{aligned} \dot{u} = \Delta u - a(x, t, u, \nabla u) + (Bv)(x, t), &\quad \text{in } Q, \\ u = 0, &\quad \text{on } \Sigma, \\ u(0) = u^0 \in L^2(\Omega). & \end{aligned} \tag{3.11}$$

Here Ω is a bounded domain of $\mathbb{R}^n$, $n \geqslant 1$; $a(x, t, u, p)$ is measurable in x, t, u, p with respect to Lebesgue measure and continuous in u, p for almost all $(x, t) \in Q$; B is a linear operator defined on a control space with range in $L^2(Q)$. One- and two-dimensional cases with specific choices of B were also discussed.

In subsequent papers, Khapalov (1998, 1999a, 1999b) studied the case where

$$(Bv)(x, t) = v(x, t)y\chi_{\mathfrak{O}}(x). \tag{3.12}$$

3.7.

Liu and Williams (1997) and Khapalov (1999c) studied the problem of exact controllability of system (3.7). The former authors proved exact controllability under the assumption that the function $f(t,u)$ appearing in $(3.7)_1$ is continuous in T on $[0,T]$ and globally Lipschitz continuous in u on $\mathbb{R}$, that is there exists a positive constant k such that

$$|f(t,z_1) - f(t,z_2)| \leqslant k|z_1 - z_2|, \text{ for all } z_1, z_2 \in \mathbb{R}. \tag{3.13}$$

Under this assumption we have the following exact controllability result.

THEOREM 9. *There exists a $T_0 > 0$ such that for $0 < T \leqslant T_0$ system (3.7) is exactly controllable in $L^2(\Omega)$ at time T, that is, for any $u^0, u_T \in L^2(\Omega)$ there exists $g(x,t) \in L^2(0,T;H^{-1}(\Omega))$ such that the solution of (3.7) with $\mathfrak{O} = \Omega$ satisfies*

$$u(x,T) = u_T, \qquad in\ \Omega. \tag{3.14}$$

Furthermore, for any fixed $u^0 \in L^2(\Omega)$, the control function

$$g(x,t;u^0,u_T) : L^2(\Omega) \to L^2(0,T;H^{-1}(\Omega))$$

is Lipschitz continuous.

The proof is based on a construction of nonlinear monotone operator.

REMARK 11. Liu and Williams (1997) claim that we cannot expect the exact internal controllability for the semilinear heat equation if $\mathfrak{O}$ is a proper subset of Ω. The results of Khapalov (1999c) do not confirm such a conjecture. This author showed that under the assumption of continuity of $f(u)$ and assuming appropriate superlinear growth at infinity of this function, the exact null-controllability in $L^2(\Omega)$ of system (3.7) is possible. More precisely, in Khapalov's approach the set $\mathfrak{O}$ depends on t, i.e.,

$$\mathfrak{O}(\cdot) = \{(x,t) | x \in \mathfrak{O}(t) \subset \Omega, \quad t \in (0,T)\} \subset Q.$$

Thus in (3.7) we have $\chi_{\mathfrak{O}}(t)$. The measure of $\mathfrak{O}(t)$ can be chosen *arbitrarily small* at all times t. The notion of exact null-controllability used by Khapalov (1999c) is a generalization of the usual one.

3.8.

Bodart and Fabre (1994) considered a semilinear heat equation with partially known initial and boundary conditions. The insensitizing problem consists in finding a control function such that some functional of the state is locally insensitive to the *perturbations of these initial and boundary conditions*. The insensitivity conditions are equivalent to a particular nonlinear exact controllability problem for parabolic equations. Since, as we already know, exact controllability is difficult to achieve, it is thus reasonable to introduce the concept of *approximately insensitizing control*, and then to solve a nonlinear approximate controllability problem of special type.

4. Thermal problems in deformable solids

In the previous section coupling between heat propagation and deformability has not been taken into account. The aim of this section is to discuss such coupled problems. Lions (1988b, Chapter 3) studied the controllability of a simplified system of thermoelasticity by the RHUM method (Reverse or Reachability Hilbert Uniqueness Method). The approximate partial controllability of the same coupled system was investigated by Glowinski and Lions (1996, Section 7), where also an optimal control problem and its dual were studied.

4.1. CONTROLLABILITY

Consider an isotropic and homogeneous thermoelastic body occupying an open and bounded set $\Omega \subset \mathbb{R}^n (n \geqslant 1)$ with boundary $\Gamma = \partial\Omega$ of class C^2. The displacement vector is denoted by $\boldsymbol{u} = (u_i(x,t))$ and the temperature by $\theta = \theta(x,t)$.

Zuazua (1995) considered the following thermoelastic system in the presence of the control $\boldsymbol{g} \in L^2(Q)^n$

$$\begin{array}{ll} \ddot{\boldsymbol{u}} - \mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla \mathrm{div}\boldsymbol{u} + \alpha\nabla\theta = \boldsymbol{g}\chi_{\mathfrak{O}}, & \text{in } Q, \\ \dot{\theta} - \Delta\theta + \beta \mathrm{div}\dot{\boldsymbol{u}} = 0, & \text{in } Q, \\ \boldsymbol{u} = \boldsymbol{0}, \quad \theta = 0, & \text{on } \Sigma, \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1, \quad \theta(0) = \theta^0, & \text{on } \Omega, \end{array} \tag{4.1}$$

where μ, λ denote Lamés constants and $\alpha, \beta > 0$ are coupling parameters; the set $\mathfrak{O}$ is the same as in the previous section. System (4.1) possesses a unique solution $(\boldsymbol{u}, \dot{\boldsymbol{u}}, \theta) \in C([0,T], H)$, where

$$H = H_0^1(\Omega)^n \times L^2(\Omega)^n \times L^2(\Omega)^n. \tag{4.2}$$

The irreversibility and the regularizing effect of the heat equation satisfied by the temperature imply that the exact controllability property may not hold. Hence, it is natural to formulate the following *exact-approximate controllability* problem: Given $(\boldsymbol{u}^0, \boldsymbol{u}^1, \theta^0)$ and $(\boldsymbol{u}_T, \boldsymbol{u}_T^1, \theta_T^0)$ and $\epsilon > 0$, find a control g such that the solution of (4.1) satisfies

$$\begin{array}{l} \boldsymbol{u}(T) = \boldsymbol{u}_T^0, \quad \dot{\boldsymbol{u}}(T) = \boldsymbol{u}_T^1, \\ \|\theta(T) - \theta_T^0\|_{L^2(\Omega)} \leqslant \epsilon. \end{array}$$

Obviously, the exact controllability would occur if $\theta(T) = \theta_T^0$.

The main result of Zuazua (1995) is formulated as follows.

THEOREM 10. *Let $\mathfrak{O}$ be a neighborhood of the boundary $\Gamma = \partial\Omega$, i.e., $\mathfrak{O} = \Omega \cap \mathfrak{O}_1$ where $\mathfrak{O}_1$ is a neighborhood of Γ in $\mathbb{R}^n$. Suppose that $T > diam(\Omega \backslash \mathfrak{O}/\sqrt{\mu}$. Then, system (4.1) is exact-approximately controllable in time T.*

The proof combines the observability inequality for the adjoint system of thermoelasticity, multiplier techniques, compactness arguments, Holmgren's Uniqueness Theorem and the result due to Henry et al. (1993) on decoupling of the thermoelasticity system.

REMARK 12. As we know, it is not natural to expect exact controllability results for the system of thermoelasticity. However, we can expect to be able to reach any *sufficiently smooth* final state, for instance the null controllability state is such a state. Lebeau and Zuazua (1998) solved this problem for simplified as well as complete system of thermoelasticity in a compact, connected C^∞ Riemannian manifolds, thus generalizing the results due to Lebeau and Robbiano (1995), cf. also Sec. 3. of our paper.

REMARK 13. The problem of partial exact (mechanical) boundary controllability was studied by Liu (1998b). The boundary condition on $\Gamma_1 \subset \Gamma$ is as follows

$$\mu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + (\lambda + \mu)(\text{div } \boldsymbol{u})\boldsymbol{n} + a(x)(\boldsymbol{m} \cdot \boldsymbol{n})\boldsymbol{u} = \boldsymbol{g}_2, \quad \text{on } \Gamma_1 \times (0, T),$$

where $\boldsymbol{g} = -(\boldsymbol{m} \cdot \boldsymbol{n})\boldsymbol{u}$ is a control. The proof of the relevant Th. 2.3 of Liu (1998b) should be read with "Correction"; this proof requires a smallness condition on α, β (the coupling parameters).

4.2. DECAY OF SOLUTIONS

Aassila (1998c) and Ouazza (1997) studied decay rates of solutions to the following simplified system of thermoelasticity with nonlinear damping

$$\begin{array}{ll} \ddot{\boldsymbol{u}} - \Delta \boldsymbol{u} + \alpha \nabla \theta + \boldsymbol{g}(\dot{\boldsymbol{u}}) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \dot{\theta} - k\Delta\theta + \beta \text{div } \dot{\boldsymbol{u}} = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \boldsymbol{u} = \boldsymbol{0}, \theta = 0 & \text{on } \Gamma \times \mathbb{R}^+, \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1, \theta(0) = \theta^0, & \text{in } \Omega. \end{array} \tag{4.3}$$

The energy of the system is defined as follows

$$E(t) = \frac{1}{2} \int_\Omega (|\dot{\boldsymbol{u}}(t)|^2 + |\nabla \boldsymbol{u}(t)|^2 + |\theta(t)|^2) dx. \tag{4.4}$$

Ouazza (1997) proved that if $\boldsymbol{g}$ is globally a Lipschitz function such that $\boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0}$ and if there exists a constant $c_1 > 0$ such that for each $\mathbf{z} \in \mathbb{R}^n$,

$$\boldsymbol{z} \cdot \boldsymbol{g}(\boldsymbol{z}) \geqslant c_1 |\boldsymbol{z}|^2, \tag{4.5}$$

then for each weak solution of (4.3)

$$E(t) \leqslant E(0)\exp(1 - \omega t), \tag{4.6}$$

where $\omega > 0$. If $\boldsymbol{g}$ is such as previously and (4.5) holds for $|\boldsymbol{z}| > 1$, and

$$\boldsymbol{z} \cdot \boldsymbol{g}(\boldsymbol{z}) \geqslant c_2 |\boldsymbol{z}|^{p+1},$$

for $|\boldsymbol{z}| \leqslant 1$, then

$$E(t) \leqslant c_3 t^{-2/(p-1)}, t > 0, \tag{4.7}$$

where $c_3 > 0$ depends on $E(0)$.

Under weaker assumptions on Ω and $\boldsymbol{g}$ Aassila (1998c) proved strong stability, i.e., $E(t) \to 0$ as $t \to \infty$ for every weak solution of (4.3).

4.2.1.

By using the methods of geometric optics (Ralston, 1982) combined with the decoupling method due to Henry et al. (1993), Lebeau and Zuazua (1999) studied the decay of solutions to the following system of thermoelasticity in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, of class C^∞

$$\begin{array}{ll} \ddot{\boldsymbol{u}} - \mu\Delta\boldsymbol{u} - (\lambda + \mu)\nabla \mathrm{div}\boldsymbol{u} + \alpha\nabla\theta = \boldsymbol{0} & \text{in } \Omega \times \mathbb{R}^+, \\ \dot{\theta} - \Delta\theta + \beta \mathrm{div}\dot{\boldsymbol{u}} = 0 & \text{on } \Omega \times \mathbb{R}^+, \\ \boldsymbol{u} = \boldsymbol{0}, \quad \theta = 0 & \Gamma \times \mathbb{R}^+, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}^0(x), \dot{\boldsymbol{u}}(x,0) = \boldsymbol{u}^1(x), \theta(x,0) = \theta^0 & \text{in } \Omega. \end{array} \tag{4.8}$$

Introduce the following condition:

Condition C. If $\varphi \in H_0^1(\Omega)^n$ is such that

$$-\Delta\boldsymbol{\varphi} = \gamma^2\boldsymbol{\varphi}, \quad \mathrm{div}\boldsymbol{\varphi} = 0; \quad \boldsymbol{\varphi} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{4.9}$$

for some $\gamma \in \mathbb{R}$, then $\boldsymbol{\varphi} = \boldsymbol{0}$. This condition was introduced in 1968 by Dafermos who proved that, see Lebeau and Zuazua (1999),

$$E(t) \to 0 \text{ as } t \to \infty$$

if and only if the domain Ω satisfies condition (C). The energy is defined by

$$E(t) = \frac{1}{2}\int_\Omega \left[\mu|\nabla\boldsymbol{u}(x,t)|^2 + (\lambda+\mu)|\mathrm{div}\boldsymbol{u}(x,t)|^2 + |\dot{\boldsymbol{u}}(x,t)|^2 + \frac{\alpha}{\beta}|\theta(x,t)|^2\right] dx. \tag{4.10}$$

It is easy to show that the energy decreases along trajectories; more precisely

$$\frac{dE(t)}{dt} = -\frac{\alpha}{\beta}\int_\Omega |\nabla\theta(x,t)|^2 dx \leqslant 0. \tag{4.11}$$

Condition (C) fails when Ω is a ball in $\mathbb{R}^n$.

The question studied by Lebeau and Zuazua (1999) concerns the exponential decay of the energy. By using the decoupling method due to Henry et al. (1993) the problem is reduced to the analysis of the Lamé system

$$\begin{aligned} \ddot{\mathbf{\Phi}} - \mu\Delta\mathbf{\Phi} - (\lambda+\mu)\nabla \mathrm{div}\mathbf{\Phi} = \mathbf{0} &\quad \text{in } Q = \Omega\times(0,T),\\ \mathbf{\Phi} = \mathbf{0}, &\quad \text{on } \Gamma\times(0,T),\\ \mathbf{\Phi}(x,0) = \mathbf{\Phi}^0(x),\ \dot{\mathbf{\Phi}}(x,0) = \mathbf{\Phi}^1(x), &\quad \text{in } \Omega. \end{aligned} \tag{4.12}$$

The general theorems on exponential (uniform) decay rate are formulated as follows

THEOREM 11. *Assume that $n = 2$ or 3. In the class of domains Ω satisfying condition (C), the exponential decay property*

$$\exists c,\omega > 0,\ E(t) \leqslant cE(0)e^{-\omega t}, \forall t > 0, \tag{4.13}$$

for the system (4.8) holds if and only if there exists $T > 0$ and $c_1 > 0$ such that

$$\|\Phi^0\|_{L^2(\Omega)^n} + |\Phi^1\|^2_{H^{-1}(\Omega)^n} \leqslant c_1 \int_0^T \|div\Phi\|^2_{H^{-1}(\Omega)} dt, \tag{4.14}$$

holds for every solution of the Lamé system (4.13).

THEOREM 12. *Assume that $\mu > 0, \lambda + 2\mu > 0$ and $\mu \neq \lambda + 2\mu$. Assume that Ω is convex or such that there exists a ray of geometric optics in Ω of arbitrarily large length which is always reflected perpendicularly on the boundary. Then the observability inequality (4.14) for the Lamé system fails for any $T > 0$ and therefore the decay of solutions of (4.8) is not uniform.*

We observe that convex domains may be classified in two sets:
(i) Those in which condition (C) fails. In this case there are solutions that do not decay as $t \to \infty$.
(ii) Those in which (C) holds. In this case every solution converges to zero but without a uniform decay rate. Convex domains are generically in the class (ii). It is not known whether the class (i) contains any convex domain other than the ball.

Lebeau and Zuazua (1999) studied also the polynomial decay of solutions of system (4.8).

4.2.2.

Liu (1998b) established a sufficient condition which guarantees the exponential decay rate of the energy by means of an additional boundary damping. The reason why the energy $E(t)$ does not tend to zero as $t \to \infty$ is that the total energy is not dissipated completely in the form of thermal energy. Therefore Liu (1998b) introduced a velocity feedback on part of the boundary of the thermoelastic body, which is clamped along the rest of the boundary, to increase the loss of energy.

The domain Ω is star-shaped and the boundary velocity feedback is assumed in the following form

$$\begin{aligned} &\theta = 0, && \text{on } \Gamma \times \mathbb{R}^+, \\ &\boldsymbol{u} = \boldsymbol{0} && \text{on } \Gamma \times \mathbb{R}^+, \\ &\mu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + (\lambda + \mu)(\mathrm{div}\boldsymbol{u})\boldsymbol{n} && \\ &+a(\boldsymbol{m} \cdot \boldsymbol{n})\boldsymbol{u} + (\boldsymbol{m} \cdot \boldsymbol{n})\dot{\boldsymbol{u}} = 0, && \text{on } \Gamma_1 \times \mathbb{R}^+, \end{aligned} \tag{4.15}$$

where $a = a(x)$ is a given nonnegative function on Γ_1, with

$$a \in C^1(\Gamma_2). \tag{4.16}$$

The methods of proofs are based on multiplier techniques and the asymptotic property of the semigroups. By using Russell's "controllability via stabilizability" principle (see Russell, 1978), Liu solved also the problem of *partial exact boundary controllability*.

4.3. THERMOVISCOELASTICITY

In this section we shall review the results due to Liu and Williams (1998) and Liu (1998a) on partial exact controllability and exponential stabilization of thermoviscoelastic systems.

4.3.1.

Consider the problem of partial exact controllability with Dirichlet boundary controls for the following system

$$\begin{aligned} &\ddot{\boldsymbol{u}} - \mu\Delta\boldsymbol{u} - (\lambda + \mu\nabla\mathrm{div}\boldsymbol{u} + \alpha\nabla\theta - \epsilon \int_0^t G(t-\tau)[\mu\Delta\boldsymbol{u}(x,\tau) && \\ &+(\lambda+\mu)\nabla\mathrm{div}\boldsymbol{u}(x,\tau)]d\tau = 0, && \text{in } Q, \\ &\dot{\theta} - \Delta\theta + \alpha\mathrm{div}\dot{\boldsymbol{u}} = 0, && \text{in } Q, \\ &\boldsymbol{u} = \boldsymbol{g}, \theta = 0, && \text{on } \Sigma, \\ &\boldsymbol{u}(x,0) = \boldsymbol{0}, \dot{\boldsymbol{u}}(x,0) = \boldsymbol{0}, \theta(x,0) = 0, && \text{in } \Omega. \end{aligned} \tag{4.17}$$

Here Ω is a star-shaped domain in $\mathbb{R}^n$ of class C^2, $G(t)$ denotes the relaxation function (see Deseri et al. 1999) and $\boldsymbol{g}$ the control acting on a part of Σ; in fact, on $\Sigma(x^0)$.

For an isotropic material the reΩxation function is an isotropic fourth-order, time dependent tensor. Liu assumes its simplified form with only one essential component.

The main result of Liu and Williams (1998) is stated as follows.

THEOREM 13. *Suppose that $G \in H^2(0,T)$ and $T > 2R(x^0)/\sqrt{\mu}$. Then there exists $\epsilon_0, \alpha_0 > 0$ such that if $\epsilon \leqslant \epsilon_0$ and $\alpha \leqslant \alpha_0$, then for every state $(\boldsymbol{u}_T, \boldsymbol{u}_T^1) \in L^2(\Omega)^n \times H^{-1}(\Omega)^n$ there exists a control $\boldsymbol{g} \in L^2(\Sigma(x^0))^n$ steering the displacement of system (4.18) to the state $(\boldsymbol{u}_T, \boldsymbol{u}_T^1)$.*

We recall that $\Sigma(x^0) = \Gamma(x^0) \times (0, T)$ and

$$\Gamma(x^0) = \{x \in \Gamma | \boldsymbol{m}(x) \cdot \boldsymbol{n}(x) > 0\}.$$

REMARK 14. (i) Boundary observability inequality (indirect inequality) was established by using classical multiplier techniques whilst the main result by the HUM.
(ii) Theorem 13 requires the coefficients α and β to be "small". The general case seems to remain an open problem.
(iii) The solution $(\boldsymbol{u}, \theta)$ of system (4.17) satisfies

$$\begin{aligned} &\boldsymbol{u} \in C([0, T], L^2(\Omega)^n) \cap C^1([0, T]), H^{-1}(\Omega)^n), \\ &\theta \in C([0, T], L^2(\Omega)). \end{aligned} \tag{4.18}$$

(iv) Liu and Williams (1998) claim, after Lagnese (1990), that it is not possible to exactly control θ by means of the boundary displacement control alone, and it is physically unrealistic to use θ as an additional control variable. However, the role of viscous effect was not revealed.

4.3.2.

Liu (1998a) studied the problem of uniform stabilization of the following thermoviscoelastic system with a boundary velocity feedback

$$\begin{aligned}
&\ddot{\boldsymbol{u}} - \mu\Delta\boldsymbol{u} - (\lambda + \mu)\nabla\mathrm{div}\boldsymbol{u} && \\
&+\mu G * \Delta\boldsymbol{u} + (\lambda + \mu)G * \nabla\mathrm{div}\boldsymbol{u} + \alpha\nabla\theta = 0, && \text{in } \Omega \times \mathbb{R}^+, \\
&\dot{\theta} - \Delta\theta + \beta\mathrm{div}\dot{\boldsymbol{u}} = 0, && \text{in } \Omega \times \mathbb{R}^+, \\
&\theta = 0, && \text{on } \Gamma \times \mathbb{R}^+, \\
&\boldsymbol{u} = \boldsymbol{0} && \text{on } \Gamma_0 \times \mathbb{R}^+, \\
&\mu\frac{\partial}{\partial \boldsymbol{n}}(\boldsymbol{u} - G * \boldsymbol{u}) + (\lambda + \mu)(\mathrm{div}(\boldsymbol{u} - G * \boldsymbol{u}))\boldsymbol{n} && \\
&+a(\boldsymbol{m} \cdot \boldsymbol{n})(\boldsymbol{u} - G * \boldsymbol{u}) + (\boldsymbol{m} \cdot \boldsymbol{n})\dot{\boldsymbol{u}} = \boldsymbol{0}, && \text{on } \Gamma_1 \times \mathbb{R}^+, \\
&\boldsymbol{u}(0) = \boldsymbol{u}^0, \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1, \theta(0) = \theta^0, && \text{in } \Gamma_1 \times \mathbb{R}^+, \\
&\boldsymbol{u}(0) - \boldsymbol{u}(-s) = \boldsymbol{w}^0(s), && \text{in } \Omega \times \mathbb{R}^+.
\end{aligned} \tag{4.19}$$

As previously the domain Ω is assumed to be star-shaped and of class C^2; we assume that $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$. The sign "$\star$" denotes the convolution product in time:

$$G * v(t) = \int_{-\infty}^{t} G(t - \tau)v(x, \tau)d\tau.$$

The relaxation function satisfies physically plausible assumptions which, in particular, imply

$$G(\infty) = \lim_{t\to\infty} G(t) = 0.$$

The function $a = a(x)$ is a given nonnegative function with $a \in C^1(\Gamma^1)$ and $\boldsymbol{w}^0(x,\tau)$ is a specified history. Furthermore, $A = \max_{x\in\Gamma_1} a(x)$ is assumed to be small enough and the following condition holds

$$\Gamma_1 \neq \emptyset \text{ or } a(x) \not\equiv 0. \tag{4.20}$$

Under the above assumptions the thermoviscoelastic energy defined by

$$E(t) = \kappa\|\boldsymbol{u}(t)\|^2_{H^1_{\Gamma_0(\Omega)^n}} + \frac{1}{2}\left[\|\dot{\boldsymbol{u}}(t)\|^2_{L^2(\Omega)^n} + \frac{\alpha}{\beta}\|\theta(t)\|^2_{L^2(\Omega)^n}\right] + \int_{-\infty}^{t} G(t-\tau)\|\boldsymbol{u}(t)-\boldsymbol{u}(\tau)\|^2_{H^1{}_{\Gamma_1}(\Omega)^n}d\tau, \tag{4.21}$$

decays exponentially

$$E(t) \leqslant cE(0)e^{-\omega t}, \quad \forall t \geqslant 0.$$

For solutions of (4.19) with $(\boldsymbol{u}^0, \boldsymbol{u}^1, \theta^0, \boldsymbol{w}^0) \in \mathcal{H}$. The positive constants can be explicitly given (the formulae are quite involved) and

$$\kappa = 1 - \int_0^\infty G(t)dt > 0.$$

The spaces are defined as follows

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma_0\},$$

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega)^n \times L^2(\Omega)^n \times L^2(\Omega) \times L^2(G,(0,\infty),H^1_{\Gamma_0}(\Omega)^n).$$

The history space $L^2(G,(0,\infty),H^1_{\Gamma_0}(\Omega)^n)$ consists of functions $\boldsymbol{w}$ on $(0,\infty)$ for which

$$\|\boldsymbol{w}\|^2_{L^2(G,(0,\infty),H^1_{\Gamma_0}(\Omega)^n)} = \int_0^\infty G(s)\|\boldsymbol{w}(\tau)\|^2_{H^1{}_{\Gamma_0}(\Omega)^n}d\tau < \infty.$$

REMARK 15. (i) The proof is based on the semigroup approach, multiplier techniques and Lyapunov methods. The Lyapunov function is of a generalized type.
(ii) Liu (1998a) discussed also weakening of the assumptions specified above. In all cases the problem of the exponential decay of the energy remains open.

5. Final remarks

Chapter 8 of the book by Panagiotopoulos (1993) summarizes the research of this author and of his coworkers on the optimal control and the parameter identification problems of systems governed by *hemivariational inequalities*. It seems that *nonconvex* analysis has not yet been incorporated in the study of exact (and approximate) controllability and stabilization problems. For instance, it would be interesting to consider nonconvex contact and interface conditions for elastic vibrating bodies.

Another challenging problem would be a study of remodelling of tissues as control problems, cf. Telega and Lekszycki (2000). It seems that in biological materials cells play the role of controls.

On account of limited number of pages of this contribution the result related to asymptotic analysis and elastic structures could not be included, cf. Telega and Bielski (1999, 2000).

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